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ON THE SCHRÖDINGER EIGENVALUE PROBLEM
II

BY

OLAVI HELLMAN

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CONTENTS

Synopsis

The present work is a continuation of the author's paper 'On the Schrödinger Eigenvalue Problem', Mat. Fys. Medd. Dan. Vid. Selsk. **32**, no. 4. (1960) 24 pp., in which the potential was assumed to be of the form $\frac{a}{r^2} - \frac{b}{r} - V_0(r)$, $V_0(r)$ possessing a power-series expansion within a finite interval and vanishing elsewhere. It was required that $\sqrt{(2l+1)^2 + 4ka}$ was not an integer, a condition which is now dropped. The eigenvalue problem is again reduced to the formation of a series whose terms are obtained by a simple process the main operation of which is the integration. The formulae contain the potential $V_0(r)$ as such. The eigenvalues appear as roots of rapidly converging power series; the eigenfunctions are expressed in terms of functions obtained in the process of forming the said power series. The method is applicable also to the case where the mass is a function of r.

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1. Introduction and Summary

In this paper we shall consider the Schrödinger eigenvalue problem

$$\left.\begin{aligned} &\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}-\frac{l(l+1)}{r^2}R+k(E-V(r))R=0\\ &R(0)<\infty \quad \text{and} \quad R(r)\to 0 \quad \text{for} \quad r\to\infty, \end{aligned}\right\} \tag{1.1}$$

with $l = 0,1, 2, \ldots$ and

$$V(r)=\frac{a}{r^2}-\frac{b}{r}-V_0(r), \tag{1.2}$$

where $V_0(r) \neq 0$ only for $r \varepsilon [0, L]$, possessing in that interval an absolutely convergent power-series expansion in r; a and b are arbitrary real constants.

In a previous paper[1] the above eigenvalue problem was considered in the special case of $l = 0$ and in the case where $\sqrt{(2\,l+1)^2+4\,ka}$ is not an integer —a condition which we shall drop here, thus including in our treatment the important case of $a = 0$.

The main result of the present paper will be the formula

$$\det\begin{pmatrix} -\frac{\varkappa}{2}A(1)+C(1) & (c-p-1-\lambda)\,W_{c,\,q}(2\,\lambda)-\left[q^2-\left(c-\frac{1}{2}\right)^2\right]W_{c-1,\,q}(2\,\lambda) \\ A(1) & W_{c,\,q}(2\,\lambda) \end{pmatrix}=0 \tag{1.3}$$

for the eigenvalue $\lambda=\sqrt{k\,|\,E\,|\,L^2}$. Here

$$\left.\begin{aligned} A(s)=e^{-\frac{\varkappa}{2}s}\Bigg\{1+\sum_{n=1}^{\infty}\int_0^{s}e^{\varkappa s_1}s_1^{-m}\,ds_1\int_0^{s_1}e^{-\varkappa s_2}s_2^{m}(\lambda^2-V(s_2))\,ds_2\int_0^{s_2}\cdots\\ \cdots\int_0^{s_{2n-1}}e^{-\varkappa s_{2n}}s_{2n}^{m}(\lambda^2-V(s_{2n}))\,ds_{2n}\Bigg\} \end{aligned}\right\} \tag{1.4}$$

and

$$C(s) = e^{\frac{\varkappa}{2}s} s^{-m} \int_0^s e^{-\frac{\varkappa}{2}s_1} s_1^m (\lambda^2 - V(s_1)) A(s_1) ds_1, \tag{1.5}$$

where $V(s) = \left(\frac{\varkappa}{2}\right)^2 + kL^2 V_0(Ls)$ and $s = \frac{r}{L}$ (see also (2.18a)). The series $A(s)$ converges absolutely for every $s \varepsilon [0, 1]$ and for every finite λ. The functions $W_{c,q}(x)$ — known as Whittaker functions—are defined by the integral formula[(3)]

$$W_{c,q}(x) = \frac{e^{-\frac{x}{2}} x^c}{\Gamma\left(\frac{1}{2} - c + q\right)} \int_0^\infty t^{-c-\frac{1}{2}+q} \left(1 + \frac{t}{x}\right)^{c-\frac{1}{2}+q} e^{-t} dt, \tag{1.6}$$

which may be used in practical calculations at least in the case where $q-c$ and $q+c$ are integers*. The constants $\varkappa$, c, m, q, and p of the formulae (1.3)—(1.6), expressed in the notation of (1.1) and (1.2), are as follows:

$$\varkappa = \frac{2\,kbL}{2(p+1)},\ c = \frac{kbL}{2\lambda},\ m = 2(p+1),\ q = \frac{m-1}{2},\ p = \sqrt{\left(l + \frac{1}{2}\right)^2 + ka} - \frac{1}{2}. \tag{1.7}$$

The eigenfunction of (1.1), corresponding to the eigenvalue λ_i, is given by

$$R_i(s) = \left\{ \begin{array}{lll} C s^{\frac{m-2}{2}} A_i(s) & \text{for} & s \varepsilon [0, 1] \\ C \sigma(\lambda_i) \frac{1}{s} W_{c,q}(2\lambda_i s) & \text{for} & s \varepsilon (1, \infty). \end{array} \right\} \tag{1.8}$$

Here, $A_i(s)$ is defined by (1.4) with λ_i substituted for λ, the factor $\sigma(\lambda_i)$ is obtained from (1.3), and C is the normalizing constant. As is seen, the eigenfunction is obtained as a by-product of our calculations.

In certain shell-model problems of nuclear physics the constant k of (1.1) appears as a function of r. In the cases where $k(r)(E-V(r))$ is of the form (1.2) our method is applicable after a straightforward modification.

The eigenvalue problem (1.1) is thus on the whole reduced to the forming of the series (1.4). The terms of that series contain the potential function $V_0(r)$ as such. Consequently the power-series expansion of $V_0(r)$ is not needed in the calculations; the mere existence of such an expansion is all

* Cf. J. Blomqvist and S. Wahlborn, Arkiv för Fysik **16** (1960) 545.

that is required for our purposes. Otherwise the method is largely independent of the form of the potential function—which may very well be tabular or graphical—and the elementary character of the analysis employed makes it particularly well suited for digital computers.

In practical applications one would calculate a finite number of terms of (1.4) for (1.5).

In section 2 the solution of (1.1) is found for $r \varepsilon [0, L]$ by the aid of a general theorem from the matrix calculus. The solution of (1.1) for $r \varepsilon [L, \infty]$ is known in closed form in terms of Whittaker functions ((1.6)). Equation (1.3) is obtained from the condition that $R(r)$ and $\frac{dR(r)}{dr}$ must be continuous at $r = L$. By way of application, the case of a square well is considered. The treatment of a more general case ($V_0(r)$ as defined on page 3) would be practically the same.

The matrix calculations of section 2 are carried out in detail in section 3, where also the convergence questions are dealt with.

2. The Derivation of the Eigenvalue Equation

We put (1.1) into matrix form as follows:

$$\frac{dz}{ds} - \left(\frac{P_{-1}}{s} + P(s)\right) z = 0, \tag{2.1}$$

where

$$P_1 = \begin{pmatrix} 0 & 0 \\ 0 & -2(p+1) \end{pmatrix}, \tag{2.2}$$

$$P(s) = \begin{pmatrix} -\frac{\varkappa}{2} & 1 \\ -\left(\frac{\varkappa}{2}\right)^2 + \lambda^2 - V_1(s) & \frac{\varkappa}{2} \end{pmatrix} \tag{2.3}$$

and

$$z = s^{-p} \begin{pmatrix} 0 & 1 \\ 1 & \frac{\varkappa}{2} - \frac{p}{s} \end{pmatrix} \begin{pmatrix} \frac{dR}{dr} \\ R \end{pmatrix}. \tag{2.4}$$

In the above formulae one has

$$\left.\begin{aligned} \varkappa &= \frac{2\,kbL}{2\,(p+1)} \\ p &= \sqrt{\left(l+\frac{1}{2}\right)^2 + ka} - \frac{1}{2} \\ \lambda^2 &= k\,|\,E\,|\,L^2 \\ V_1(s) &= kL^2\,V_0(Ls)\,. \end{aligned}\right\} \qquad (2.5)$$

The properties of matrix differential equations of the kind (2.1) in which $P(s)$ possesses a power-series expansion in s converging absolutely for every s within a finite interval $[0, s_1]$ are set out in a theorem by Gantmacher[2]. It follows from this theorem, since the matrix $P(s)$ of (2.1) possesses the required power-series expansion, that the general solution of (2.1) will be of the form

$$z = A\,(s)\,s^{P_{-1}}\,s^{U}\,z_0 \qquad (2.6)$$

with z_0 an arbitrary vector. Here

$$U = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}, \qquad (2.6\,\mathrm{a})$$

where α is a constant that depends on P_{-1} and $P(s)$, vanishing if $2\,(p+1)$ in P_{-1} is not an integer. The matrix $A(s)$ is regular at $s = 0$ so that $A(0) = I$. Consequently there exists a convergent power-series expansion

$$A\,(s) = I + \sum_{k=1}^{\infty} A_k s^k. \qquad (2.7)$$

We shall now calculate the matrix $A(s)$ by a procedure, similar to that employed in ref. [1], which obviates the use of power-series expansions (cf. for instance ref. [2]).

Substitution of (2.6) in (2.1) yields for A the equation

$$\frac{dA}{ds} + \frac{1}{s}\,(AP_{-1} - P_{-1}A) - P(s)\,A + s^{m-1}\,AU = 0\,, \qquad (2.8)$$

where m stands for $2\,(p+1)$. As is seen on direct substitution, (2.8) is satisfied for every $s\,\varepsilon\,[s_0, 1]$, $s_0 > 0$, by the matrix

$$A = s^{P_{-1}}\,\tilde{\Omega}\,[s^{-P_{-1}}\,P(s)\,s^{P_{-1}}] \begin{pmatrix} 1 & -\ln s^{\alpha} \\ 0 & 1 \end{pmatrix} s^{-P_{-1}}, \qquad (2.9)$$

where $B = s^{-P_{-1}} P(s) s^{P_{-1}}$. The matrix $\tilde{\Omega}(B)$ is defined as follows[1]:

$$\tilde{\Omega}(B) = I + \int^s B ds_1 + \int^s B ds_1 \int^{s_1} B ds_2 + \int^s B ds_1 \int^{s_1} B ds_2 \int^{s_2} B ds_3 + \cdots, \qquad (2.10)$$

where, in carrying out the integrations, we put the constants of integration equal to zero in all terms.

We shall prove in section 3 that A, as defined by (2.9), may be expanded into a power series in s which converges absolutely for every $s \varepsilon [0, 1]$ and has the value I for $s = 0$, i.e., we shall prove that the expression (2.9) is really the $A(s)$-matrix of the general solution of the matrix differential equation (2.1).

From (2.4) and (2.9) it now follows that

$$\left.\begin{aligned} \begin{pmatrix} \frac{dR}{dr} \\ R \end{pmatrix} &= s^{p-1} \begin{pmatrix} -\frac{\varkappa}{2}s + p & s \\ s & 0 \end{pmatrix} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & s^{-m} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 1 & -\ln s^{\alpha} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & s^m \end{pmatrix} \right\} \times \\ &\times \begin{pmatrix} 1 & 0 \\ 0 & s^{-m} \end{pmatrix} \begin{pmatrix} 1 & \ln s^{\alpha} \\ 0 & 1 \end{pmatrix} z_0, \end{aligned}\right\} \qquad (2.11)$$

where we have written briefly

$$\tilde{\Omega}(s^{-P_{-1}} P(s) s^{P_{-1}}) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \qquad (2.12)$$

The elements A, B, C, and D are calculated in section 3 (cf. eqs. (3.12), (3.13)). Our $\begin{pmatrix} \frac{dR}{dr} \\ R \end{pmatrix}$ is required to be regular at the origin. Now, the expression in braces has this property, and besides $p \geqslant 1$; consequently the vector

$$\begin{pmatrix} 1 & 0 \\ 0 & s^{-m} \end{pmatrix} \begin{pmatrix} 1 & \ln s^{\alpha} \\ 0 & 1 \end{pmatrix} z_0 = \begin{pmatrix} 1 & \ln s^{\alpha} \\ 0 & s^{-m} \end{pmatrix} z_0 \qquad (2.13)$$

must be regular at the origin. This implies the following form for z_0:

$$z_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} C_1, \qquad (2.14)$$

where C_1 is a constant. For $s \neq 0$ we obtain from (2.11) and (2.14)

$$\begin{pmatrix} \frac{dR}{dr} \\ R \end{pmatrix} = s^{p-1} \begin{pmatrix} s & p \\ 0 & s \end{pmatrix} \begin{pmatrix} -\frac{\varkappa}{2} A + s^{-m} C \\ A \end{pmatrix} C_1. \qquad (2.15)$$

The vector (2.15) is now the general solution of (2.1) for every $s \varepsilon [0,1]$ which is regular at the origin. As the constant α has disappeared, we are spared the tedious task of calculating its value.

The solution of (2.1) for $s \varepsilon (1, \infty)$ such that $\begin{pmatrix} \frac{dR}{dr} \\ R \end{pmatrix} \to 0$ for $s \to \infty$ is, according to ref.[1],

$$\begin{pmatrix} \frac{dR}{dr} \\ R \end{pmatrix} = \frac{C_2}{s^3} \begin{pmatrix} s & p \\ 0 & s \end{pmatrix} \begin{pmatrix} (c-p-1-\lambda s)\, W_{c,q}(2\lambda s) - \left[q^2 - \left(c - \frac{1}{2}\right)^2\right] W_{c-1,q}(2\lambda s) \\ s W_{c,q}(2\lambda s) \end{pmatrix}; \qquad (2.16)$$

here C_2 is an arbitrary constant, and the function $W_{c,q}(x)$ is defined by (1.6). The constants p, $\varkappa$, m, c, and q are given by (1.7) and (2.5). Since the vectors (2.15) and (2.16) must be equal for $s = 1$, it is necessary that

$$\det \begin{pmatrix} -\frac{\varkappa}{2} A(1) + C(1) & (c-p-1-\lambda)\, W_{c,q}(2\lambda) - \left[q^2 - \left(c - \frac{1}{2}\right)^2\right] W_{c-1,q}(2\lambda) \\ A(1) & W_{c,q}(2\lambda) \end{pmatrix} = 0, \qquad (2.17)$$

where, according to (2.12), (3.8), (3.9), and (3.13),

$$\left.\begin{aligned} A(s) = e^{-\frac{\varkappa}{2}s} \Big\{ 1 + \\ + \sum_{n=1}^{\infty} \int_0^s e^{\varkappa s_1} s_1^{-m} ds_1 \int_0^{s_1} e^{-\varkappa s_2} s_2^m F(s_2)\, ds_2 \int_0^{s_2} \cdots \int_0^{s_{2n-1}} e^{-\varkappa s_{2n}} s_{2n}^m F(s_{2n})\, ds_{2n} \Big\} \end{aligned}\right\} \qquad (2.18)$$

and

$$\left.\begin{aligned} C(1) = e^{\frac{\varkappa}{2}} \int_0^1 e^{-\varkappa s} s^m F(s) \Big\{ 1 + \\ + \sum_{n=1}^{\infty} \int_0^s e^{\varkappa s_1} s_1^{-m} ds_1 \int_0^{s_1} e^{-\varkappa s_2} s_2^m F(s_2)\, ds_2 \int_0^{s_2} \cdots \int_0^{s_{2n-1}} e^{-\varkappa s_{2n}} s_{2n}^m F(s_{2n})\, ds_{2n} \Big\} ds. \end{aligned}\right\} \qquad (2.19)$$

We summarize the symbols used in (2.17)—(2.19):

$$\left.\begin{aligned}
F(s) &= -\left(\frac{\varkappa}{2}\right)^2 + \lambda^2 - kL^2 V_0(Ls) \\
\varkappa &= \frac{2\,kbL}{m} \\
m &= 2(p+1) \\
c &= \frac{2\,bL}{2\,\lambda} \\
\lambda^2 &= k\,|E|\,L^2 \\
p &= \sqrt{\left(l+\frac{1}{2}\right)^2 + ka} - \frac{1}{2}.
\end{aligned}\right\} \quad (2.20)$$

For practical calculations it is convenient first to expand (2.18) into a power series in λ^2. This we are allowed to do because (2.18) converges absolutely for every $s\,\varepsilon\,[0,1]$ (cf. section 3). As would be easy to show, we obtain

$$A(s) = 1 + \sum_{k=1}^{\infty}\sum_{r=k}^{\infty}\lambda^{2k}(-1)^{r+k}\sum(r;k)\int_0^s f ds_1 \int_0^{s_1} h ds_2 \int_0^{s_2} \dots \int_0^{s_{2r-1}} g ds_{2r}$$

with

$$f = e^{\varkappa s} s^{-m}, \quad g = e^{-\varkappa s} s^m \left[\left(\frac{\varkappa}{2}\right)^2 + V_1(s)\right] \quad \text{and} \quad h = e^{-\varkappa s} s^m.$$

The term

$$\sum(r;k)\int_0^s f ds_1 \int_0^{s_1} h ds_2 \int_0^{s_2} \dots \int_0^{s_{2r-1}} g ds_{2r}$$

has the following meaning: k g's out of the r g's of the integral

$$\sum(r;0)\int_0^s f ds_1 \int_0^{s_1} \dots \int_0^{s_{2r-1}} g ds_{2r} = \int_0^s f ds_1 \int_0^{s_1} g ds_2 \int_0^{s_2} f ds_3 \int_0^{s_3} \dots f ds_{2r-1} \int_0^{s_{2r-1}} g ds_{2r}$$

are replaced by h, and the summation is extended over the integrals obtained on the performance of all permutations of these k h's and $r-k$ g's. For instance,

$$\sum(2;1)\int_0^s f ds_1 \int_0^{s_1} h ds_2 \int_0^{s_2} f ds_3 \int_0^{s_3} g ds_4$$

$$= \int_0^s f ds_1 \int_0^{s_1} g ds_2 \int_0^{s_2} f ds_3 \int_0^{s_3} h ds_4 + \int_0^s f ds_1 \int_0^{s_1} h ds_2 \int_0^{s_2} f ds_3 \int_0^{s_3} g ds_4.$$

As is clear from (2.15) and (2.16),

$$R_i(s) = \begin{cases} C_1 s^p A_i(s) & \text{for } s\varepsilon[0,1] \\ C_1 \sigma(\lambda_i)\dfrac{1}{s} W_{c,q}(2\lambda s) & \text{for } s\varepsilon(1,\infty], \end{cases} \tag{2.21}$$

the factor $\sigma(\lambda_i)$ being obtained from (2.17). The constant C_1 is obtained from a normalizing condition.

For an application we shall consider the simplest example, the square well. It is clear from the above formulae that the treatment of a more general case would be essentially the same. Now

$$a = b = 0$$

and

$$V_0(r) = \begin{cases} V_0 & \text{for } r\varepsilon[0, L] \\ 0 & \text{for } r\varepsilon(L, \infty]. \end{cases}$$

According to (2.20) we have

$$\varkappa = 0, \quad m = 2(l+1), \quad c = 0, \quad \text{and} \quad q = l + \frac{1}{2}.$$

We then obtain from (2.17)—(2.19)

$$\det\begin{pmatrix} \dfrac{(\lambda^2-\lambda_0^2)}{m+1} + \displaystyle\sum_{n=1}^{\infty} \dfrac{(\lambda^2-\lambda_0^2)^{n+1}}{2^n n!(m+2n+1)!!} & -(1+l+\lambda)W_{0,q}(2\lambda) - \left(q^2-\dfrac{1}{4}\right)W_{-1,q}(2\lambda) \\ 1 + \displaystyle\sum_{n=1}^{\infty} \dfrac{(\lambda^2-\lambda_0^2)^n}{2^n n!(m+2n-1)!!} & W_{0,q}(2\lambda) \end{pmatrix} = 0, \tag{2.22}$$

where $\lambda^2 = k|E|L^2$, $\lambda_0^2 = kV_0L^2$ and $(m+2n\pm1)!! = \prod_{k=0}^{n}(m+2k\pm1)$. The functions $W_{0,l+\frac{1}{2}}(x)$ and $W_{-1,l+\frac{1}{2}}(x)$ are calculated most conveniently by application of (1.6).

In case $m = 4$, we have, as is readily seen,

$$\det\begin{pmatrix} \left(\sqrt{\lambda_0^2-\lambda^2}\right)^2 \sin\sqrt{\lambda_0^2-\lambda^2} + 3\sqrt{\lambda_0^2-\lambda^2}\cos\sqrt{\lambda_0^2-\lambda^2} - 3\sin\sqrt{\lambda_0^2-\lambda^2} & \lambda^2+3\lambda+3 \\ -\sqrt{\lambda^2-\lambda^2}\cos\sqrt{\lambda_0^2-\lambda^2} + \sin\sqrt{\lambda_0^2-\lambda^2} & -\lambda-1 \end{pmatrix} = 0. \tag{2.23}$$

By using the power-series expansions of $\sin\sqrt{\lambda_0^2-\lambda^2}$ and $\cos\sqrt{\lambda_0^2-\lambda^2}$ and the formula (1.6) it is easy to show that (2.23) and (2.22), with $m = 4$, are identical.

3. The Form and Properties of the Matrix $\tilde{\Omega}[s^{-P_{-1}} P(s) s^{P_{-1}}]$

We shall first establish a useful property of the matrices $\tilde{\Omega}$. *Lemma*: Let us assume that it is possible to write the (square) matrix H defined in the interval $[0,1]$ and singular at the origin in the form $H = H_1 + H_2$ so that the elements of the matrix H_1 are integrable over the interval $[0,1]$ and those of H_2 are integrable over every interval $[s_0, 1]$ with $0 < s_0 < 1$. Then, for $s \varepsilon [s_0, 1]$,

$$\tilde{\Omega}(H_1 + H_2) = \Omega_0^s(H_1)\,\tilde{\Omega}\left\{[\Omega_0^s(H_1)]^{-1} H_2 \Omega_0^s(H_1)\right\}, \tag{3.1}$$

where

$$\Omega_0^s(H_1) = I + \int_0^s H_1\, ds_1 + \int_0^s H_1\, ds_1 \int_0^{s_1} H_1\, ds_2 + \cdots.$$

As is well known[2], the formula (3.1) holds in the case of $s_0 = 0$.

Proof: Since the elements of H_1 are integrable over the interval $[0,1]$, the matrizant $\Omega_0^s(H_1)$ exists for every $s \varepsilon [0, 1]$, and so does $[\Omega_0^s(H_1)]^{-1}$ (cf. ref.[2]). Consequently the matrix $[\Omega_0^s(H_1)]^{-1} H_2\, \Omega_0^s(H_1)$ has the same properties as the matrix H_2. We now proceed by the same method as that applied to the case of $s_0 = 0$ in ref.[2]. Let us put

$$X = \Omega_0^s(H_1), \qquad Y = \tilde{\Omega}(H_1 + H_2)$$

and $Y = XZ$. By differentiation we obtain

$$(H_1 + H_2)Y = H_1 XZ + X\frac{dZ}{ds}$$

for every $s \varepsilon [s_0, 1]$, $0 < s_0 < 1$, from which it follows that

$$\frac{dZ}{ds} = X^{-1} H_2 XZ = [\Omega_0^s(H_1)]^{-1} H_2 \Omega_0^s(H_1)\, Z.$$

Hence

$$Z = \tilde{\Omega}\left\{[\Omega_0^s(H_1)]^{-1} H_2 \Omega_0^s(H_1)\right\}$$

for all $s \varepsilon [s_0, 1]$, $0 < s_0 < 1$, which completes the proof.

As was shown in ref.[1], the series $\tilde{\Omega}(B)$, where B is a matrix singular at $s = 0$, converges absolutely for every $s \varepsilon [s_0, 1]$ when $0 < s_0 < 1$.

Remembering (2.2) and (2.3), we then obtain

$$s^{-P_{-1}} P(s)\, s^{P_{-1}} = \begin{pmatrix} -\frac{\varkappa}{2} & s^{-m} \\ F(s)\, s^m & \frac{\varkappa}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\varkappa}{2} & 0 \\ 0 & \frac{\varkappa}{2} \end{pmatrix} + \begin{pmatrix} 0 & s^{-m} \\ F(s)\, s^m & 0 \end{pmatrix},$$

where m stands for $2(p+1)$ and $F(s)$ for $-\left(\frac{\varkappa}{2}\right)^2+\lambda^2-V_1(s)$. We can now identify H_1 and H_2 of our lemma with

$$\begin{pmatrix} -\frac{\varkappa}{2} & 0 \\ 0 & \frac{\varkappa}{2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & s^{-m} \\ F(s)\, s^m & 0 \end{pmatrix}$$

respectively. Then

$$\left.\begin{aligned} \Omega_0^s(H_1) &= \begin{pmatrix} e^{-\frac{\varkappa}{2}s} & 0 \\ 0 & e^{\frac{\varkappa}{2}s} \end{pmatrix}, \\ [\Omega_0^s(H_1)]^{-1} &= \begin{pmatrix} e^{\frac{\varkappa}{2}s} & 0 \\ 0 & e^{-\frac{\varkappa}{2}s} \end{pmatrix} \end{aligned}\right\} \tag{3.2}$$

and

$$[\Omega_0^s(H_1)]^{-1} H_2\, \Omega_0^s(H_1) = \begin{pmatrix} 0 & e^{\varkappa s} s^{-m} \\ e^{-\varkappa s} s^m F(s) & 0 \end{pmatrix}. \tag{3.3}$$

By an obvious modification of a formula introduced in ref. [4] we obtain for $s\,\varepsilon\,[s_0, 1]$, $0<s_0<1$,

$$\tilde{\Omega}\left\{[\Omega_0^s(H_1)]^{-1} H_2\, \Omega_0^s(H_1)\right\} = \begin{pmatrix} \omega_{11}(s) & \int^s e^{\varkappa s_1} s_1^{-m}\, \omega_{22}(s_1)\, ds_1 \\ \int^s e^{-\varkappa s_1} s_1^m F(s_1)\, \omega_{11}(s_1)\, ds_1 & \omega_{22}(s) \end{pmatrix} \tag{3.4}$$

with

$$\omega_{ii}(s) = \sum_{n=0}^{\infty} \alpha_{2n}^{(i)}(s), \qquad i = 1,2, \tag{3.5}$$

$$\left.\begin{aligned} &\alpha_{2n}^{(1)}(s) = \\ &\int^s e^{\varkappa s_1} s_1^{-m}\, ds_1 \int^{s_1} e^{-\varkappa s_2} s_2^m F(s_2)\, ds_2 \int^{s_2} \cdots \int^{s_{2n-1}} e^{\varkappa s_{2n-1}} s_{2n-1}^{-m}\, ds_{2n-1} \int^{s_{2n-1}} e^{-\varkappa s_{2n}} s_{2n}^{-m} F(s_{2n})\, ds_{2n} \end{aligned}\right\} \tag{3.6}$$

and

$$\left.\begin{aligned} &\alpha_{2n}^{(2)}(s) = \\ &\int^s e^{-\varkappa s_1} s_1^m F(s_1)\, ds_1 \int^{s_1} e^{\varkappa s_2} s_2^{-m}\, ds_2 \int^{s_2} \cdots \int^{s_{2n-1}} e^{-\varkappa s_{2n-1}} s_{2n-1}^m F(s_{2n-1})\, ds_{2n-1} \int^{s_{2n-1}} e^{\varkappa s_{2n}} s_{2n}^{-m}\, ds_{2n}, \end{aligned}\right\} \tag{3.7}$$

where $\alpha_0^{(1)}(s) = \alpha_0^{(2)}(s) = 1$*.

* The formula (3.4) may be verified also by direct differentiation after the absolute convergence of the power series $\omega_{11}(s)$ and that of the power series $\alpha_{22}^{(1)}(s)$ contained in $\omega_{22}(s)$ (cf. (3.10)) have been proved.

The series $\omega_{11}(s)$ now converges absolutely for every $s\varepsilon[0,1]$ (i.e., not only for $s\varepsilon[s_0,1]$). Indeed, from the fact that, for $s\varepsilon[0,1]$,

$$|F(s)| < M < \infty, \quad e^{\varkappa s} \leqslant e^{\varkappa} \text{ and } e^{-\varkappa s} \leqslant 1,$$

it follows that

$$|\alpha_{2n}^{(1)}(s)| < (e^{\varkappa} M)^n \frac{s^{2n}}{2^n n!(m+1)(m+3)\dots(m+2n-1)}$$

for every $n = 0, 1, 2, \dots$ and for every $s\varepsilon[0,1]$.

As both $e^{\pm\varkappa s}$ and $F(s)$ possess a power-series expansion in s absolutely convergent for every $s\varepsilon[0,1]$, the same is true of $\omega_{11}(s)$. Since it was required that $A(0) = I$, and since $\omega_{11}(s)$ is regular at $s = 0$ and $\Omega_0^0(H_1) = I$, we must replace (3.6) by

$$\alpha_{2n}^{(1)}(s) = \int_0^s e^{\varkappa s_1} s_1^{-m} ds_1 \int_0^{s_1} e^{-\varkappa s_2} s_2^m F(s_2) ds_2 \int_0^{s_2} \dots \int_0^{s_{2n-1}} e^{-\varkappa s_{2n}} s_{2n}^m F(s_{2n}) ds_{2n}. \tag{3.8}$$

As $e^{-\varkappa s} s^m F(s)$ is continuous for every $s\varepsilon[0,1]$, and the series $\omega_{11}(s)$ is absolutely continuous for every $s\varepsilon[0,1]$, the integral $\int_0^s e^{-\varkappa s_1} s_1^m F(s_1)\,\omega_{11}(s_1)\,ds_1$ exists for every $s\varepsilon[0,1]$ and has the properties of $s^{m+1}\omega_{11}(s)$. In compliance with the requirement that $\omega_{21}(0) = 0$, we set

$$\omega_{21}(s) = \int_0^s e^{-\varkappa s_1} s_1^m F(s_1)\,\omega_{11}(s_1)\,ds_1. \tag{3.9}$$

The series $\omega_{22}(s)$ is more complicated since the integrations will produce an $\ln s$—function in each $\alpha_{2n}^{(2)}(s)$. As may be shown by means of the absolutely convergent power-series expansions of $e^{\pm\varkappa s}$ and $F(s)$, the functions $\omega_{22}(s)$ and $\omega_{12}(s)$ have the respective forms

$$\omega_{22}(s) = \left(\sum_{k=1}^{m} Q_k\right)\left\{\int_0^s e^{-\varkappa s_1} s_1^m F(s_1)\,\omega_{11}(s_1)\,ds_1\right\}\cdot \ln s + \omega_{22}^{(2)}(s) \tag{3.10}$$

and

$$\omega_{12}(s) = \left(\sum_{k=1}^{m} Q_k\right)\omega_{11}(s)\cdot \ln s + s^{-m+1}\omega_{22}^{(1)}(s), \tag{3.11}$$

where the lower limits of the integrals have been chosen to meet the requirement $A(0) = I$. The Q_k are numbers depending on $\varkappa$, m and $F(s)$. The functions $\omega_{22}^{(1)}(s)$ and $\omega_{22}^{(2)}(s)$ have the properties of $\omega_{11}(s)$; in particular, $\omega_{22}^{(1)}(0) = \omega_{22}^{(2)}(0) = 1$.

We now have from (3.1), (3.2) and (3.4)

$$\left.\begin{array}{c}\tilde{\Omega}\left(s^{-P_{-1}} P s^{P_{-1}}\right)= \\ \begin{pmatrix} e^{-\frac{\varkappa}{2} s}\,\omega_{11}(s) & e^{-\frac{\varkappa}{2} s}\left[\left(\sum_{k=1}^{m} Q_k-\alpha\right)\omega_{11}(s)\ln s+s^{-m+1}\,\omega_{22}^{(2)}(s)\right] \\ e^{\frac{\varkappa}{2} s}\int_0^s e^{-\varkappa s_1} s_1^m F(s_1)\,\omega_{11}(s_1)\,ds_1 & e^{\frac{\varkappa}{2} s}\left[\left(\sum_{k=1}^{m} Q_k-\alpha\right)\left\{\int_0^s e^{-\varkappa s_1} s_1^m F\omega_{11}\,ds_1\right\}\ln s+\omega_{22}^{(1)}(s)\right]\end{pmatrix},\end{array}\right\} \tag{3.12}$$

$\omega_{11}(s)$ being defined by (3.5) and (3.8). By choosing $\alpha=\sum_{k=1}^{m} Q_k$ we obtain from (2.9) and (3.12)

$$A(s)=\begin{pmatrix} e^{-\frac{\varkappa}{2} s}\,\omega_{11}(s) & e^{-\frac{\varkappa}{2} s}\,s\,\omega_{22}^{(2)}(s) \\ e^{\frac{\varkappa}{2} s}\,s^{-m}\int_0^s e^{-\varkappa s_1} s_1^m F(s_1)\,\omega_{11}(s_1)\,ds_1 & e^{\frac{\varkappa}{2} s}\,\omega_{22}^{(1)}(s)\end{pmatrix}. \tag{3.13}$$

Since the functions $\omega_{22}^{(1)}(s)$ and $\omega_{22}^{(2)}(s)$ are not needed in the calculations, their forms are not given here. As was made clear above, the matrix $A(s)$ has all the required properties: $A(s)$ may be expanded into a power series in s convergent in the interval $[0,1]$, and A has the property $A(0)=I$. From the fact that $A(s)$ satisfies (2.8) for every $s\,\varepsilon\,[s_0, 1]$, $0<s_0<1$, it now follows that $A(s)$ satisfies (2.8) for every $s\,\varepsilon[0, 1]$.

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